Scaling and dimensional analysis

Examples from the living world

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What is scaling?

Not all physical and chemical phenomena are affected in the same way by a change in length scale.

When the size is changed, the structure of system has also to be changed to serve a given function
We have an intuitive understanding of scaling

Suharris1262’s picture on Flickr

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Godzilla, Mothra, Rodan and Ghidorah!
Walking is a complex problem

- At least 6 degrees of freedom for each leg
  - 2 for the ankle
  - 1 for the knee
  - 3 for the hip

- About 29 important muscles in each leg
Dimensional analysis of walking

What are the main variables relevant to walking?

- The velocity
  \( v \) \( [v] = \text{L T} \)
- The length of the leg
  \( l \) \( [l] = \text{L} \)
- The animal weight
  \( m \) \( [m] = \text{M} \)
- Gravity
  \( g \) \( [g] = \text{L T}^{-2} \)
- The stride length
  \( s \) \( [s] = \text{L} \)
- ...

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Buckingham’s $\Pi$ theorem

$n = 5$ parameters  \quad n-k = 2$
$k = 3$ dimensions  \quad \text{dimensionless groups}

The analysis of walking takes the simple form $Y = f(X)$
Dimensional analysis of animal locomotion

Taking $v$ and $s$ as core variables, the two dimensionless numbers are found to be:

$$\Pi_1 = \frac{s}{l}$$
$$\Pi_2 = \frac{v^2}{lg}$$

Froude number

William Froude (1810-1879)
Real animals do obey dynamical scaling

\[ \frac{\text{stride length}}{\text{leg length}} \]

**FIGURE 2** A graph of (stride length/leg length) against the square root of Froude number for humans and various animals. Each animal is represented by several points, recording observations of it moving at different speeds. The square root of Froude number is plotted, rather than Froude number itself, to avoid undue clumping of points in the lower range of speeds.

1 quadruped = 2 bipeds
Japanese monsters do not follow the scaling law

![Graph showing stride length/leg length against the square root of Froude number for humans and various animals.](image)

**FIGURE 2** A graph of (stride length/leg length) against the square root of Froude number for humans and various animals. Each animal is represented by several points, recording observations of it moving at different speeds. The square root of Froude number is plotted, rather than Froude number itself, to avoid undue clumping of points in the lower range of speeds.
Dinosaurs ichnites

\( \Pi_1 = \text{s/l} \) is known for many extinct species

- Brontosaurus (30 tonnes) 1 m/s
- Tyrannosaurus (5 tonnes) 2 m/s
- « Smaller ones » (0.5 tonnes) 12 m/s

250 Ma ago

Chirotherium

Laetoli footprints (3.6 Ma ago)

Fossil footprints discovered in Tanzania in 1976.

Assuming that they were left by Australopithecus afarensis, the speed was estimated to be 1 m/s.

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Dimensionless groups are ratios of orders of magnitude

\[ Froude = \frac{mv^2}{mgl} = \frac{Kinetic\ Energy}{Gravitational\ Energy} \]

It is physically impossible to walk at Froude > 1
Why is it difficult to walk on the moon?

For the same $v$ and $l$, the Froude is 6 times larger on the moon. It is hard not to be in the running range.

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There are as many different scaling laws as there are physical phenomena

- Dynamic equilibrium of the body
- Static equilibrium of the body
- Metabolism
- …

The problem that is analyzed is determined by the choice of the dimensional variables
How does the diameter of the legs scale with the weight?
Dimensional analysis of leg diameter

What are the main variables relevant to leg diameter?

- The diameter of the leg $d$ with $[d] = L$
- The animal weight $m$ with $[m] = M$
- Gravity $g$ with $[g] = L \, T^{-2}$
- The maximum stress that the leg can withstand $\sigma$ with $[\sigma] = M \, L^{-1} \, T^{-2}$
Scaling of the diameter of a leg

\[ \Pi = \frac{\sigma d^2}{mg} = \text{constant} \]

\[ d \approx \sqrt{\frac{mg}{\sigma}} \]

A single dimensionless number

Galileo Galilei (1564-1642)
Progress in King Kong movies

Merian Cooper, Ernest Schoedsack (1933)  
Peter Jackson (2005)
Qualitative aspects of scaling: the respiratory system

The respiratory system of insects is (mostly) passive; the size of insects is therefore limited by oxygen diffusion.
Dimensional analysis of insect breathing

What are the main variables relevant to insect breathing?

- The size of the insect: $d$ with $[d] = L$
- Diffusion coeff of $O_2$: $D$ with $[D] = M^2T^{-1}$
- $O_2$ concentration: $C$ with $[C] = \text{Mol L}^{-3}$
- Insect metabolism: $r_V$ with $[r_V] = \text{Mol L}^{-3} T^{-1}$
Buckingham’s Π theorem

\[ n = 4 \text{ variables} \quad \Rightarrow \quad n-k = 1 \]
\[ k = 3 \text{ dimensions} \quad \Rightarrow \quad \text{dimensionless group} \]

\[ \frac{r_v d^2}{D C_0} \]

This is the insect’s Thiele modulus.
Insects can live only if that number is small.
Qualitative aspects of scaling: the respiratory system

The giant insects of the Carboniferous could exist only because the oxygen concentration in the atmosphere was about 30-35%.


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Summing up...

1. Understand the physics
   
   This is a key step: what are the relevant variables?

2. Data reduction with dimensionless groups (Buckingham’s $\Pi$)
   
   Identify dimensionless groups. Similarity is achieved when dimensionless groups take the same value

3. Dimensionless groups are ratios of orders of magnitude
   
   With some experience, you can start the analysis directly at this point, and jump from 1 to 3.

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All this is universal!
Further reading

  Very thorough: Π theorem, its justification and applications.

- J.B.S. Haldane, On being the right size
  Beautiful essay about why animals have different sizes

- R. McNeill Alexander, Walking and Running,
  The mathematical gazette, 80 (1996) 262
  http://www.jstor.org/pss/3619558
  A very clear and entertaining paper by the biologist who estimated first the walking speeds of dinosaurs.