Two problems
on dimensional analysis

Cedric J. Gommes

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Contents

A few portraits 1

1 Back to basics: dimensional analysis 5
   1.1 Rayleigh’s method of dimensions a.k.a. Buckingham’s theorem 5
   1.2 The educated art of dimensional analysis . . . . . . . . . . . . 7

2 Problems 10
   2.1 The power of the Trinity nuclear test . . . . . . . . . . . . . . 10
   2.2 Time-dependent evaporation from a pond in quiescent air . . . 14
Figure 1: Galileo Galilei (1564-1632) painted here by il Tintoretto, was an Italian astronomer/mathematician/physicist who is widely acknowledged as a founder of modern science, in particular for his use of experimentation. He was also one of the first to understand the concept of scaling.

Figure 2: This is a drawing from Galileo’s *Dialogues Concerning Two New Sciences*. “I refer especially to his last assertion which I have always regarded as a false, though current, opinion; namely, that in speaking of these and other similar machines one cannot argue from the small to the large, because many devices which succeed on a small scale do not work on a large scale”.

The complete text can be found [here](#).
Figure 3: John William Strutt, third baron Rayleigh (1842-1919) was a prolific British physicist. He is famous for many discoveries, but the reason for his presence in this chapter is his formulation and use of the “method of dimensions”, which is nowadays better known as Buckingham’s Π theorem.

Figure 4: Osbourne Reynolds (1842-1912) is an Irish engineer who gave his name to what is probably the most famous of all dimensionless numbers. Incidentally, Reynolds seems to be the first person in the UK to hold the title of “Professor of Engineering” (at the university of Manchester).
Figure 5: John Burdon Sanderson Haldane (1892-1964), here around 1900, was a prominent British-born naturalised Indian biologist. He notably contributed to genetics, physiology and evolutionary biology. He was also a prolific writer of popular science essays. The reason why I am citing him here is his delightful essay entitled “On Being The Right Size”, which you can download [here].
1 Back to basics: dimensional analysis

1.1 Rayleigh’s method of dimensions a.k.a. Buckingham’s theorem

The idea behind dimensional analysis is that a quantitative law of physics, engineering, etc. cannot depend on the particular units used to express that law. This can be put very formally, but we will stick to a simple example and generalise it afterwards.

Imagine you have a pipe of radius $R$ and length $L$, filled with small beads of diameter $d$, in which you push a given liquid so as to ensure a volumetric flow $Q$ (See Fig. 6). You are interested in the pressure $\Delta P$ necessary to reach that flow. A priori, you may expect that the viscosity $\eta$ and the density $\rho$ of the liquid will play a role in that relation. You therefore expect a general relation of the type

$$\Delta P = f(d, Q, \eta, R, L, \rho)$$  \hspace{1cm} (1)

In that relation, the set of variables $d$, $Q$, and $\eta$ can be thought of as a basis for all dimensions. In particular, we can use them to put the remaining variables in the following dimensionless form

$$\Pi_0 = \frac{d^3}{(Q\eta)} \times \Delta P$$
$$\Pi_1 = \frac{1}{d} \times R$$
$$\Pi_2 = \frac{1}{d} \times L$$
\[ \Pi_3 = \left( \frac{Q}{d\eta} \right) \times \rho \quad (2) \]

Imagine you knew the function \( f() \) in Eq. 1, either by experiment or by solving complicated equations. Using the dimensionless variables, you would be able to rewrite it as

\[ \Pi_0 = f'(d, Q, \eta, \Pi_1, \Pi_2, \Pi_3) \quad (3) \]

where naturally \( f'() \) is a different function than \( f() \).

Now here is the crux of dimensional analysis. Imagine you had obtained the function \( f() \) or \( f'() \) using the SI units (\( kg, m, s \)) and your data is well described by Eq. 3. If you change your unit of mass, say you now use pounds instead of kilograms, none of your dimensionless variables would change in value; \( d \) and \( Q \) would also keep their values because their dimensions do not contain the mass. However, \( \eta \) would change, because it would now be expressed in \( lb \ m^{-1} \ s^{-1} \) instead of \( kg \ m^{-1} \ s^{-1} \). In other words, you multiply the numerical value of \( \eta \) by a factor slightly larger than 2. Look at Eq. 3 and think about this: how is it possible that you multiply \( \eta \) by 2, leave all the other variables unchanged, and yet the equation is still satisfied? The only conclusion you can reach is that the function \( f'() \) cannot depend explicitly on \( \eta \).

You can now make the same reasoning, assuming that you change your time unit form \( s \) to \( min \). Nothing would change except the value of \( Q \) that would be multiplied by 60. And yet the Eq. 3 would still have to be satisfied. The conclusion is the same as before: \( f'() \) cannot depend on \( Q \). At this stage, the only dimensional variable left in \( f'() \) is \( d \). If you changed your unit of length form \( m \) to inches... You get it. The final conclusion is that the dependence has to be of the type

\[ \Pi_0 = f(\Pi_1, \Pi_2, \Pi_3) \quad (4) \]

where only dimensionless quantities are present. Any other form would be equivalent to accepting that the results of an experiment would depend on the particular units used to express them. This would be ridiculous.

Of course the reasoning we did is very general. Imagine you are interested in knowing how a given quantity, say \( Q_0 \) depends on a series of other quantities relevant to your system, say \( Q_1 \) to \( Q_N \). In other word, you are interested in

\[ Q_0 = f(Q_1, Q_2, \cdots, Q_N) \quad (5) \]
Among all your variables you have $k$ different dimensions, and we assume that the first $k$ quantities $Q_1, Q_2, \ldots, Q_k$ form a basis for these dimensions. Using the base variables, you can therefore form dimensionless numbers as

$$\Pi_0 = Q_0 \times Q_1^{\alpha_1} Q_2^{\alpha_2} \cdots Q_k^{\alpha_k}$$  \hspace{1cm} (6)$$

where the exponents $\alpha_1$ to $\alpha_k$ are chosen in such a way that $\Pi_0$ is dimensionless. In the same way, you can form $N - k$ other dimensionless numbers: $\Pi_1$ from $Q_{k+1}$, until $\Pi_{N-k}$ from $Q_N$.

Imagine that the function $f()$ of Eq. (5) were known to you. Plugging these dimensionless numbers in it, you would be able to rewrite it as

$$\Pi_0 = f'(Q_1, \ldots, Q_k, \Pi_1, \ldots, \Pi_{N-k})$$  \hspace{1cm} (7)$$

where $f'$() is of course a different function than $f()$. The argument that the result should not depend on the particular set of units leads you to the conclusion that $f'$() cannot depend explicitly on $Q_1, Q_2, \ldots, Q_k$. The dependence has to be

$$\Pi_0 = f(\Pi_1, \ldots, \Pi_{N-k})$$  \hspace{1cm} (8)$$

This is all there is to dimensional analysis.

1.2 The educated art of dimensional analysis

The choice of a particular set of variables to start a dimensional analysis is absolutely central. Starting from the wrong set of variables would be as flawed as solving the wrong equations! Moreover, dimensional analysis is often the occasion to make simplifying assumptions. This generally consists in making educated guesses on which variable can be neglected, or on specific combinations of variables that should be considered. Let us make this more practical with our example of liquid in a pipe.

What dimensional analysis has taught us so far is that the dependence of the pressure drop on the other relevant quantities has to be of the type

$$\frac{\Delta P d^5}{Q \eta} = f\left(\frac{R}{d}, \frac{L}{d}, \frac{\rho Q}{d \eta}\right)$$  \hspace{1cm} (9)$$

Is this useful? Honestly, it isn’t.

Let us see if we can make some of these educated guesses to obtain some more useful results. In a situation such as in Fig. 6, where $d \ll L$ and $d \ll R$, C.J. Gommes, March 6, 2014
it seems clear that the relevant quantity we should be interested in is $\Delta P/L$ and not $L$. If you double the length of the pipe, you expect the pressure drop to double. This would be like putting two pipes in series. Moreover the independent variable should not be $Q$ but the velocity $U = Q/(\pi R^2)$. Here again, if you double both $Q$ and $\pi R^2$ you do not expect the pressure drop to change at all. This would be like putting two pipes in parallel. In other words, the problem would be better formulated as

$$\frac{\Delta P}{L} = f(d, U, \eta, \rho) \quad (10)$$

There are now only five variables and still three dimensions, i.e. two dimensionless variables. We leave it as an exercise to you to prove that the relation can be put as

$$\frac{d^2 \Delta P}{\eta UL} = f \left( \frac{\rho U d}{\eta} \right) \quad (11)$$

where the argument of the function is nothing but a Reynolds number. This is already much more informative than Eq. 9.

To understand that Eq. (11) is almost all there is to know about the beads-in-the-pipe flow, you have to recall from your lectures in fluid mechanics what the Reynolds number means. A convenient way to put it here is

$$\text{Reynolds} = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{\rho U^2}{\eta U/d} \quad (12)$$

The numerator is Bernoulli’s pressure and the denominator is the viscous pressure $\eta$ times a velocity gradient.

Now for small Reynolds numbers, the dominant forces are the viscous forces. This means that the fluid density $\rho$ should have no influence on the pressure drop. The relation is therefore

$$\frac{\Delta P}{L} = f(0) \frac{\eta U}{d^2} \quad (13)$$

where $f(0)$ is in this context just a constant, which you can safely assume to be of order unity. What this result tells you is that the pressure drop scales like $1/d^2$. This is a very strong dependence! If you divide the size of the beads by two you need a pressure that is four times larger to maintain the same flow.
Figure 7: Overall shape of the function \( f(x) \) that appears in Eq. 11.

The other limit is that of large Reynolds numbers. In that case the dominant force is inertia. In that limit, the viscosity should play no role in the pressure drop. This means that the asymptotic form of \( f() \) should be

\[
f(x) \simeq x \quad \text{for} \quad x \gg 1
\]

or

\[
\frac{\Delta P}{L} = \text{constant} \times \frac{\rho U^2}{d}
\]

where once again the constant can be safely assumed to be of order unity. The transition between the two regimes corresponding to Eqs. 13 and 15 occurs for \( \rho Ud/\eta \simeq 1 \).

Summing up, dimensional analysis complemented by some educated guessing taught us

1. that the general form of the pressure drop is that of Eq. 11, so that all there is to know is the function \( f(x) \);

2. that the overall shape of the function is as shown in Fig. 7.

The actual values of \( f(x) \) naturally depend on specific features such as the packing density of the spheres, etc. so that \( f(x) \) can in general only be determined experimentally. It is particularly interesting to note that Eq. 11 provides us with several ways of measuring \( f(x) \). You can change the velocity \( U \), the size of the beads \( d \). Even more interestingly, you can determine the effect of changing the size of the beads by simply flowing a different liquid in the tube.
2 Problems

2.1 The power of the Trinity nuclear test

Figure 8: Released pictures of the Trinity nuclear test held in New Mexico on July 16, 1945. The pictures are at times $t = 3.3$ ms, $4.6$ ms, $15$ ms, and $62$ ms after detonation. On the first picture, the 150 m high pole on which the bomb was placed is still visible.

On July 16, 1945, the first atomic bomb ever was detonated in New Mexico. The pictures shown in Fig. 8 were released and published in Life Magazine. The energy of the blast, however, was highly classified and it was kept secret. The story goes that Geoffrey Ingram Taylor, the British physicist, used dimensional analysis to estimate the latter energy from the data available in the pictures. The purpose of the present exercise is to guide you through Taylor’s analysis.

1. What is seen in the pictures is a spherical shock wave separating the undisturbed air from the region affected by the explosion. As usual, a dimensional analysis is simplified by some educated guess. Taylor’s analysis is based on the following assumptions

   • the explosion itself is so rapid that the only relevant characteristic
of the bomb is the amount of energy $E$ that it releases. The duration of the explosion is irrelevant:

- the shock wave propagation is so quick that it can be modelled as an adiabatic process, characterised by adiabatic exponent $\gamma$;
- the pressure generated by the shock is much larger than the atmospheric pressure, so that the latter should not be accounted for in the analysis. Only the density of the air $\rho_0$ matters.

Based on these simplifying assumptions, use dimensional analysis to find the way in which the radius $R$ of the shock wave increases with time $t$.

2. Table 1 gathers the values of $R$ as a function of $t$, determined from the released pictures of the test. Assume that any unknown quantity in the answer to point 1 is of the order unity, use those numbers to estimate the energy of the bomb. Express that energy in equivalent mass of TNT, with conversion factor $1 \text{ g}_{\text{TNT}} = 4184 \text{ J}$.

3. No building can resist a shock wave with an amplitude of 1 bar. Use whatever method you find fit to estimate the radius of complete destruction of the Trinity test.

<table>
<thead>
<tr>
<th>time (ms)</th>
<th>3.3</th>
<th>4.6</th>
<th>15</th>
<th>62</th>
</tr>
</thead>
<tbody>
<tr>
<td>radius (m)</td>
<td>59</td>
<td>67</td>
<td>106</td>
<td>185</td>
</tr>
</tbody>
</table>

Table 1: Time-dependent radius of the Trinity shock wave, as determined from the released pictures.
Solution

1. In a dimensional form, the relation we look for is of the type

\[ R = f(t, E, \rho, \gamma) \]  \hspace{1cm} (16)

That is 5 variables and three dimensions, i.e. two dimensionless numbers. One such number is the adiabatic exponent \( \gamma \). The remaining variables can be combined as

\[ \Pi_0 = \frac{\rho R^5}{Et^2} \]  \hspace{1cm} (17)

The general solution is therefore of the type

\[ \frac{\rho R^5}{Et^2} = f(\gamma) \]  \hspace{1cm} (18)

which predicts a propagation of the shock according to a \( R \approx t^{2/5} \) power law. This law is extremely well followed by the data (see Fig. 9).

![Figure 9: Comparison of the data from the released pictures (dots) with the \( R \approx t^{2/5} \) power law (red line).](image)
2. A simple way to estimate $E \times f(\gamma)$ is obtained by calculating $\rho R^5/t^2$ from the data in Tab. 1. The values are $6.6 \times 10^{13}$ J, $6.4 \times 10^{13}$ J, $5.9 \times 10^{13}$ J and $5.6 \times 10^{13}$ J. Assuming the value $6 \times 10^{13}$ J, this corresponds to an energy of 14 kt (kilotons) of TNT. Strictly this is not the value of $E$, it is the value of $E \times f(\gamma)$ and the best we can do here is assume $f(\gamma) \simeq 1$.

The actual value of $E$ was closer to 20 kt. Still this estimate is remarkable close to the reality, given the crudeness of our analysis.

3. We should estimate first how the pressure varies with the radius of the shock, i.e.

$$P = f(R, E, \rho, \gamma)$$

(19)

Dimensional analysis tells us that the dependence should be

$$P = \frac{E}{R^3} f(\gamma)$$

(20)

where again $f(\gamma)$ is an unknown constant. Assuming it is equal to one, one estimates that the radius where the amplitude of the shock is equal to 1 bar is

$$R \simeq \left( \frac{6 \times 10^{13} \text{J}}{10^5 \text{Pa}} \right)^{1/3} = 840 \text{ m}$$

(21)

This is of course a crude estimate, but the order of magnitude is correct. In the case of Hiroshima bombing (with a power similar to the Trinity test) the reported radius of complete destruction is about 1600 m.
2.2 Time-dependent evaporation from a pond in quiescent air

In this problem, you are asked to use dimensional analysis to analyse the time-dependent rate of evaporation from a finite pond. Before you start working on this problem, it might be useful for you to study carefully the problem we did on the evaporation over an infinite surface (problem 2, [here]).

![Figure 10: Circular pond with radius R, in cylindrical coordinates (r, z).](image)

The pond is idealised as being circular with radius $R$ (see Fig. 10). We are interested in knowing how the evaporation rate depends on the size of the pond and on time.

1. Use dimensional analysis to determine the general expression for $\dot{N}$ (e.g. in mol/s) as a function of time and of the other relevant variables. Comment on the $R$-dependence of $\dot{N}$ for a given time.

2. Write down the equation that would have to be solved to determine the water vapour concentration $c(r, z, t)$ together with its boundary conditions. Put that equation and boundary conditions in dimensionless form. Find the general expression for $\dot{N}$ as a function of $c(r, z, t)$.

3. Make some educated guesses and find the asymptotic values of $\dot{N}$ for long and short times $t$. 

C.J. Gommes, March 6, 2014
Solution

1. The dimensional dependence is

\[ \dot{N} = f(t, R, D, c_0) \]  

(22)

where \( c_0 \) is the saturating concentration (i.e. on the surface of the pond). That is 5 variables and 3 dimensions, i.e. two dimensionless numbers. A convenient base set is \( R, D \) and \( c_0 \). The two dimensionless numbers are

\[ \Pi_0 = \frac{\dot{N}}{DRc_0} \quad \text{and} \quad \Pi_1 = \frac{Dt}{R^2} \]  

(23)

And the sought relation is

\[ \frac{\dot{N}}{DRc_0} = f \left( \frac{Dt}{R^2} \right) \]  

(24)

The \( R \)-dependence of \( \dot{N} \) for a given time is counter-intuitive. The evaporation rate is not proportional to the area of the pond! It is proportional to its radius.

2. Fick’s second law takes the form

\[ \partial_t c = D \left( \frac{1}{r} \partial_r (r \partial_r c) + \partial_z^2 c \right) \]  

(25)

with boundary conditions

\[
\begin{align*}
    c(r, z, t) &= 0 \quad \forall z \text{ and for } r \to \infty \\
    c(r, z, t) &= 0 \quad \forall r \text{ and for } z \to \infty \\
    \partial_r c(r, z, t) &= 0 \quad \forall z \text{ and for } r = 0 \\
    c(r, z, t) &= c_0 \quad \text{for } z = 0 \text{ and } r \leq R \\
    \partial_z c(r, z, t) &= 0 \quad \text{for } z = 0 \text{ and } r > R
\end{align*}
\]  

(26)

which define a well-posed mathematical problem.

To put these equations in a dimensionless form, the simplest option consists in defining the following dimensionless variables

\[
\bar{c} = c/c_0 \quad \bar{r} = r/R \quad \bar{z} = z/R \quad \bar{t} = (Dt)/R^2
\]  

(27)
Fick’s law takes the form

$$\partial_t \bar{c} = \frac{1}{\bar{r}} \partial_r (\bar{r} \partial_r \bar{c}) + \partial_{\bar{z}}^2 \bar{c}$$

(28)

with boundary conditions

$$\bar{c} = 0 \quad \forall \bar{z} \text{ and for } \bar{r} \to \infty$$

$$\bar{c} = 0 \quad \forall \bar{r} \text{ and for } \bar{z} \to \infty$$

$$\partial_r \bar{c} = 0 \quad \forall \bar{z} \text{ and for } \bar{r} = 0$$

$$\bar{c} = 1 \quad \text{for } \bar{z} = 0 \text{ and } \bar{r} \leq 1$$

$$\partial_{\bar{z}} \bar{c} = 0 \quad \text{for } \bar{z} = 0 \text{ and } \bar{r} > 1$$

(29)

It is important that you realise that Eq. 28 and the boundary conditions 29 do not depend on any dimensionless parameter. This means that the solution for the concentration depends only on the dimensionless variables $\bar{r}$, $\bar{z}$ and $\bar{t}$. In other words

$$c(r, z, t) = c_0 f \left( \frac{r}{R}, \frac{z}{R}, \frac{Dt}{R^2} \right)$$

(30)

where $f()$ is some unknown function.

The evaporation rate is calculated as an integral over the surface of the diffusive flux, i.e. as

$$\dot{N} = \int_0^R 2\pi r dr \quad D \left[ \partial_z c \right]_{z=0}$$

(31)

Put in front of it a minus sign if you want the evaporation rate to be positive (whatever sign convention you are happy with).

Now if you knew the

$$\frac{\dot{N}}{DRc_0} = 2\pi \int_0^1 \bar{r} d\bar{r} \quad \left[ \partial_z f \right] (\bar{r}, \bar{z} = 0, \bar{t})$$

(32)

Note that this is compatible with the general form that you obtained at point 1.

3. The educated guess is always an interesting part, in which you can exert your creativity. Perhaps you can find different ways of doing this than me.
The long-time limit is the simplest of the two. If you wait long enough, the concentration converges towards a stationary state. You could in principle imagine two different situations. Either the concentration will end up being uniform, with $C = C_0$ everywhere, in which case the limit is $\dot{N} = 0$. Or there will still be a concentration gradient and the limit of $\dot{N}$ will be a constant.

In other words, do you think that the problem is closer to the case of the infinite surface () or to the case of the evaporating droplet? In the former case, you would have $\dot{N} = 0$ and in the latter you would calculate $\dot{N}$ from $Sh=2$. Clearly, it is the second case. We therefore have

$$\dot{N} \sim DRC_0$$ \hfill (33)

For the short-time limit, you first have to realise that small-$t$ is equivalent to large-$R$. This is because the argument of the unknown function $f()$ in Eq. 24 is $Dt/R^2$. What we are actually interested in is the behaviour of $f()$ for small values of its arguments.

Having said that, the large-$R$ behaviour is a problem that we solved long ago. For very large values of $R$ the present problem is equivalent to the evaporation from an infinite surface. In that case, you know that the concentration profile depends only on the reduced variable $z/\sqrt{Dt}$. The evaporation flux (per unit area) therefore scaled like

$$J_D \sim DC_0/\sqrt{Dt}$$ \hfill (34)

The same relation is expected to hold for our present case in the limit $t \ll R^2/D$. The relation that we look for is simply obtained by multiplying this by the area of the pond. This leads to

$$\dot{N} \sim \pi R^2 DC_0/\sqrt{Dt}$$ \hfill (35)

Note that this equation can be written as

$$\dot{N} \sim DRC_0 \times \frac{\pi}{\sqrt{Dt/R^2}}$$ \hfill (36)

which is indeed of the same form as Eq. 24. We have therefore shown that $f(x) \sim 1/\sqrt{x}$ for $x \ll 1$. Note that using the exact solution for the evaporation over an infinite surface (the error function), you should be able to find the exact value of the constant in front of the $1/\sqrt{x}$.

C.J. Gommes, March 6, 2014 17